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Critical Magnetic Prandtl Number for Small-Scale Dynamo

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We report a series of numerical simulations showing that the critical magnetic Reynolds number $\mathrm{Rm_c}$ for the nonhelical small-scale dynamo depends on the Reynolds number Re. Namely, the dynamo is shut down if the magnetic Prandtl number $\mathrm{Pr_m} = \mathrm{Rm/Re}$ is less than some critical value $\mathrm{Pr_{m,c}} \lesssim 1$ even for Rm for which dynamo exists at $\mathrm{Pr_m} \geq 1$. We argue that, in the limit of $\mathrm{Re} \to \infty$, a finite $\mathrm{Pr_{m,c}}$ may exist. The second possibility is that $\mathrm{Pr_{m,c}} \to 0$ as $\mathrm{Re} \to \infty$, while $\mathrm{Rm_c}$ tends to a very large constant value inaccessible at current resolutions. If there is a finite $\mathrm{Pr_{m,c}}$, the dynamo is sustainable only if magnetic fields can exist at scales smaller than the flow scale, i.e., it is always effectively a large-$\mathrm{Pr_m}$ dynamo. If there is a finite $\mathrm{Rm_c}$, our results provide a lower bound: $\mathrm{Rm_c} \gtrsim 220$ for $\mathrm{Pr_m} \leq 1/8$. This is larger than Rm in many planets and in all liquid-metal experiments.

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The simplest description of a conducting fluid is in terms of equations of magnetohydrodynamics (MHD):

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = \nu\Delta\mathbf{u} - \nabla p + \mathbf{B}\cdot\nabla\mathbf{B} + \mathbf{f}, \quad (1)$$
$$\partial_t \mathbf{B} + \mathbf{u}\cdot\nabla\mathbf{B} = \mathbf{B}\cdot\nabla\mathbf{u} + \eta\Delta\mathbf{B}, \quad (2)$$

where $\mathbf{u}$ is velocity, $\mathbf{B}$ is magnetic field, $\mathbf{f}$ is the external-force density, ν is viscosity and η is magnetic diffusivity. The pressure gradient ∇p is determined by the incompressibility condition $\nabla\cdot\mathbf{u} = 0$. We have rescaled p and $\mathbf{B}$ by ρ and $(4\pi\rho)^{1/2}$, respectively (ρ is density).

A fundamental property of Eqs. (1–2) is the ability of the velocity and magnetic fields to exchange energy. In three-dimensional turbulent flows (and in many other chaotic flows), this can take the form of net amplification of magnetic field with time, a process referred to as MHD dynamo. There are two kinds of dynamo. The first is the mean-field dynamo (growth of $\langle\mathbf{B}\rangle$), which usually requires a net flow helicity [1]. It is a large-scale effect that must be considered in conjunction with such system-specific properties as geometry, rotation, mean shear etc. The second kind is the small-scale dynamo: amplification of the magnetic energy $\langle B^2\rangle$ due to random stretching of the field by the turbulent flow, requiring no net helicity [2, 3, 4]. The stretching is opposed by the resistive diffusion, so the dynamo is only possible when the magnetic Reynolds number $\mathrm{Rm} = \langle u^2\rangle^{1/2}\ell_0/\eta$ exceeds a certain critical value (ℓ_0 is the system scale).

In this Letter, we study the existence of small-scale dynamo in homogeneous incompressible turbulence with magnetic Prandtl number $\mathrm{Pr_m} = \nu/\eta < 1$ (i.e., $\mathrm{Re} = \langle u^2\rangle^{1/2}\ell_0/\nu > \mathrm{Rm}$). This is an important issue because $\mathrm{Pr_m}$ is small in stars ($\mathrm{Pr_m} \sim 10^{-2}$ at base of the Sun's convection zone), planets ($\mathrm{Pr_m} \sim 10^{-5}$ [5]), and in liquid-metal laboratory dynamos [6, 7, 8].

In three dimensions, most types of turbulence at scales much smaller than the system size are predominantly vortical and well described by Kolmogorov's dimensional theory [9]. In this theory, the fastest field stretching is done by the small-scale velocities. The essential physics of small-scale dynamo should thus be contained within our homogeneous, isotropic, incompressible model.

The small-scale dynamo is most transparent in the limit of $\mathrm{Pr_m} \gg 1$. Straightforward estimates show that, while velocity is dissipated at the viscous scale $\ell_\nu \sim \mathrm{Re}^{-3/4}\ell_0$, magnetic field can occupy smaller scales down to the resistive $\ell_\eta \sim \mathrm{Pr_m^{-1/2}}\ell_\nu$. The dynamo is driven by the fastest eddies: the viscous-scale ones, which are spatially smooth. The growing fields are organized in folds, with direction reversals at the resistive scale ℓ_η and field lines remaining approximately straight up to the flow scale ℓ_ν [10, 11, 12, 13]. Why this is a winning configuration is best seen on the example of a linear velocity field (locally uniform rate of strain) [3, 11]: the field aligns with the stretching direction of the flow but reverses along the "null" direction, so that compression cannot lead to resistive annihilation of antiparallel fields canceling the effect of stretching. For this mechanism to apply, it is essential that (i) the flow be spatially smooth, so fluid trajectories separate exponentially in time leading to exponential stretching, (ii) a scale separation between the field scale (reversals) and the flow be achievable. The large-$\mathrm{Pr_m}$ turbulent dynamo satisfies both conditions, as do all deterministic chaotic dynamos [4]. Thus, the small-scale dynamo, as it is usually understood, is essentially the large-$\mathrm{Pr_m}$ dynamo. The often simulated case of $\mathrm{Pr_m} = 1$ belongs to the same class: the magnetic energy is amplified at scales somewhat smaller than the viscous scale and the field structure is similar to the case of $\mathrm{Pr_m} \gg 1$ [13]. When $\mathrm{Pr_m} < 1$, the field

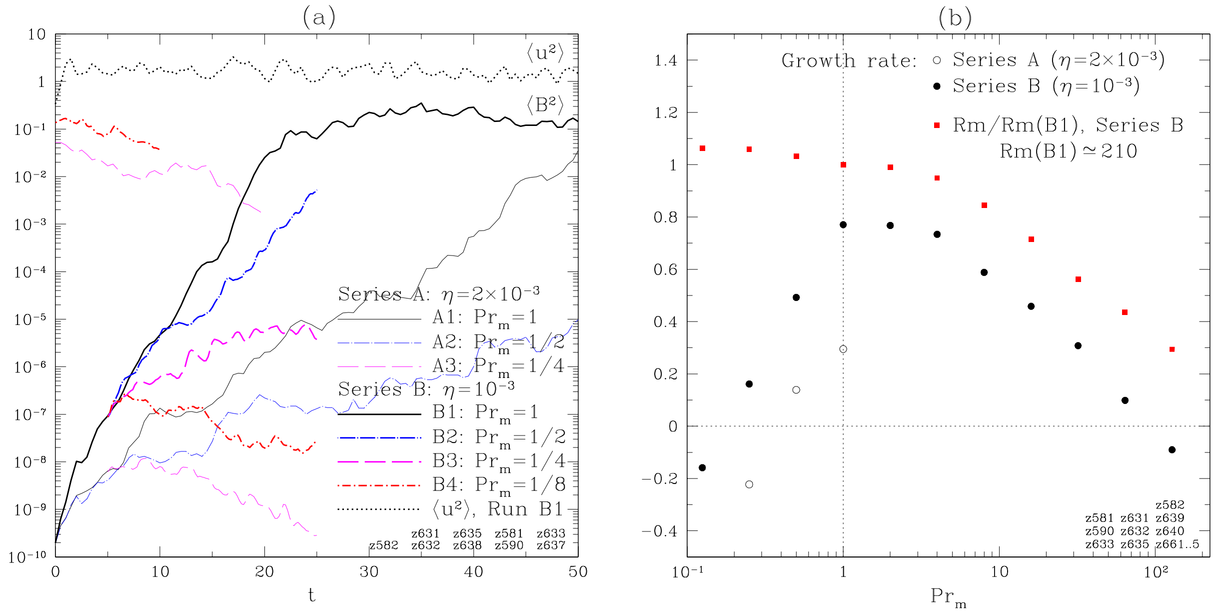

FIG. 1: (a) Evolution of magnetic energy $\langle B^2 \rangle(t)$ in all runs. Also shown are the versions of Runs A3 and B4 that started from the saturated state of Run A1 (at $t = 75$) and Run B1 (at $t = 50$), respectively. Runs A1, A2, and B1 have resolution 128^3, Runs A3, B2, B3, and B4 are 256^3. (b) Growth rates vs. $\mathrm{Pr_m}$ (at fixed η) for the same runs plus for 7 runs (128^3) extending Series B to $\mathrm{Pr_m} > 1$. The dynamo is again shut down at $\mathrm{Pr_m} \sim 100$ because velocity is strongly damped by large viscosity and Rm drops below critical. We show Rm (divided by its value for Run B1, $\mathrm{Rm} \simeq 210$) in the same plot.

scale is resistively limited to be comparable to, or larger than, the viscous cutoff. The field interacts with inertial-range motions, which are spatially rough and cannot be thought of as having a locally uniform rate of strain. Is there still a small-scale dynamo?

In order to address this question, we have carried out a series of numerical simulations. Equations (1–2) were solved in a triply periodic box by the pseudospectral method. We used a random nonhelical forcing $\mathbf{f}$ applied at the box scale and white in time. The average injected power $\epsilon = \langle \mathbf{u}\cdot\mathbf{f} \rangle$ was kept fixed. The code units are based on box size 1 and $\epsilon = 1$. Defining $\mathrm{Rm} = \langle u^2 \rangle^{1/2}/\eta k_0$ with $k_0 = 2\pi$ the box wave number, we have found that the dynamo existed for $\mathrm{Pr_m} \geq 1$ provided $\mathrm{Rm} \gtrsim 80$ (cf. Fig. 1b). Once the existence of the dynamo at $\mathrm{Pr_m} = 1$ for a given η was ascertained, η was fixed. The viscosity ν was then decreased. The results are summarized in Fig. 1. In runs with $\eta = 2 \times 10^{-3}$ (series A, $\mathrm{Rm} \simeq 110$), starting from weak field, the dynamo persisted at $\mathrm{Pr_m} = 1/2$, but was shut down at $\mathrm{Pr_m} = 1/4$. In runs with $\eta = 10^{-3}$ (series B, $\mathrm{Rm} = 220 \pm 10$), there was dynamo at $\mathrm{Pr_m} = 1/2$ and 1/4, but not at $\mathrm{Pr_m} = 1/8$. Figure 2 shows the time-averaged normalized magnetic-energy spectra for series A and B, as well as velocity spectra multiplied by k^2. The latter characterize the turbulent rate of strain and peak at the viscous scale. We see that as this scale drops below the resistive scale, the dynamo shuts down. Note that there is no initial-condition dependence: Runs A3 and B4, which decay starting from weak field, also decay if initialized in the saturated state of their $\mathrm{Pr_m} = 1$ counterparts (Fig. 1a).

Our main result is, thus, that $\mathrm{Pr_{m,c}}$ exists even as Rm is kept approximately fixed at a value for which small-scale dynamo is possible at larger $\mathrm{Pr_m}$ (Fig. 1b). In other words, *the critical magnetic Reynolds number for growth* $\mathrm{Rm_c}$ *increases with* Re. Because of resolution constraints, we cannot afford a parameter scan to produce the dependence $\mathrm{Rm_c(Re)}$. What *a priori* statements about this dependence can be made on physical grounds?

Consider first the asymptotic case $\mathrm{Re} \gg \mathrm{Rm} \gg 1$. The resistive scale then lies in the inertial range, $\ell_0 \gg \ell_\eta \gg \ell_\nu$. As, in Kolmogorov turbulence, $u_\ell/\ell \sim \ell^{-2/3}$, most of the stretching is done by the eddies at the resistive scale $\ell_\eta \sim \mathrm{Rm}^{-3/4}\ell_0$, where stretching is of the same order as diffusion [14, 15]. Since the inertial range is self-similar, the existence of the dynamo should not depend on the exact location of ℓ_η, and it is the local (in k space) properties of the turbulent velocity that determine its propensity to amplify magnetic energy. Therefore, if the dynamo fails, it does so at all $\mathrm{Pr_m}$ below some critical value $\mathrm{Pr_{m,c}}$ of order unity. The effective transition from the "large-$\mathrm{Pr_m}$" to the "small-$\mathrm{Pr_m}$" regime occurs at $\mathrm{Pr_m} = \mathrm{Pr_{m,c}}$. In this case, $\mathrm{Rm_c/Re} \to \mathrm{Pr_{m,c}} =$ const < 1 as $\mathrm{Re} \to \infty$. Thus, if our results are asymptotic, then the turbulent small-scale dynamo is always, in essence, a large-$\mathrm{Pr_m}$ one, and the folded direction-reversing fields are the only type of magnetic fluctuations that can be self-consistently generated and sustained by

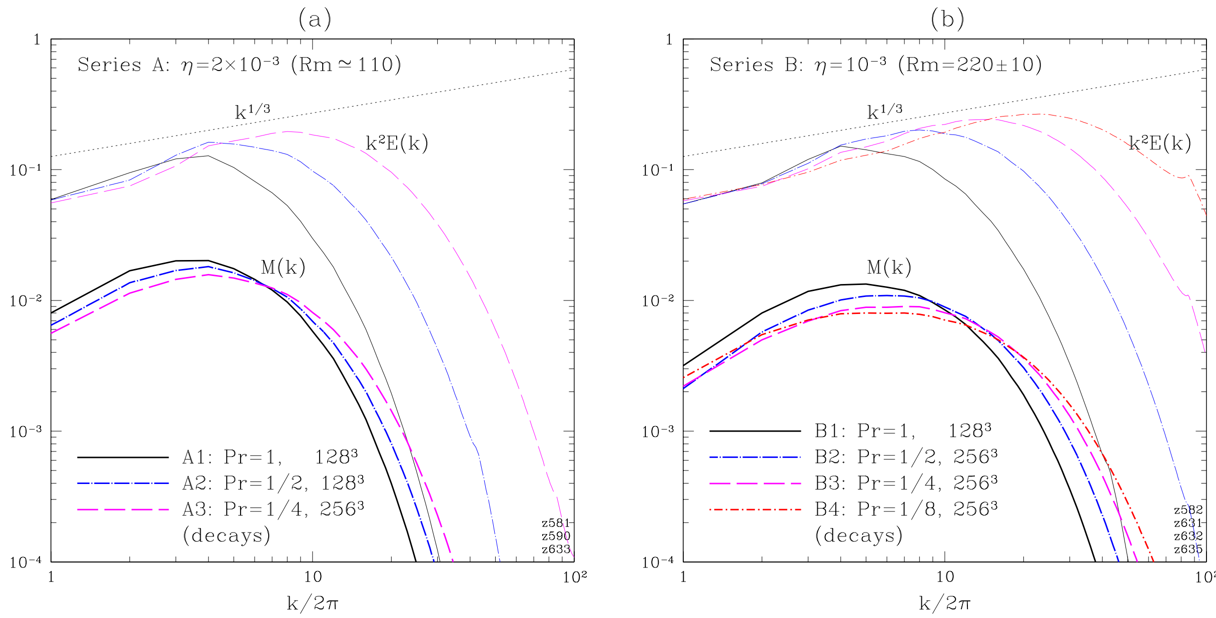

FIG. 2: Magnetic-energy spectra normalized by $\langle B^2\rangle/2$ and averaged over time: (a) Series A, (b) Series B. Also given are velocity spectra multiplied by k^2 and the reference Kolmogorov slope $k^{1/3}$. The time intervals used for averaging are for Runs A1, A2: $5 \le t \le 40$; for Run B1: $2.5 \le t \le 17.5$; for Runs A3, B2, B3, B4: $10 \le t \le 25$.

nonhelical turbulence. As the separation between parallel and transverse scales of the field diminishes at $\mathrm{Pr_m} < 1$ (Fig. 3), no steady fluctuation level can be maintained.

The second possibility is that $\mathrm{Rm_c}$ asymptotes to some constant value for Re above those we are able to resolve: $\mathrm{Rm_c} \to \mathrm{const} \gtrsim 220$ and $\mathrm{Pr_{m,c}} \to 0$ as $\mathrm{Re} \to \infty$. Our results do not rule out this outcome, whereby asymptotically in Rm and Re, the dynamo persists at low $\mathrm{Pr_m}$, but very large Rm is needed to achieve it in practice (numerically or experimentally). In stellar convective zones, Rm is, indeed, very large ($10^6 ... 10^9$ for the Sun). On the other hand, in planets and in laboratory dynamos, $\mathrm{Rm} \sim 10^2$, which is comparable to Rm in our simulations.

Note that the arguments above assume scale invariance of the inertial range, i.e., neglect the effects of intermittency. An intermittent velocity field will exhibit large coherent fluctuations of the rate of strain, which might be locally effective in stretching the magnetic field in a way similar to the large-$\mathrm{Pr_m}$ dynamo [25]. Whether these fluctuations can provide enough stretching on the average to make a workable dynamo cannot be settled qualitatively. Note that an intermittent growth by rare strong bursts is evident in $\mathrm{Pr_m} < 1$ runs where the dynamo is suppressed but not shut down (most vividly in Run A2, see Fig. 1a).

No theory of the dynamo shutdown at low $\mathrm{Pr_m}$ exists at present. Invoking turbulent diffusion (mixing) of magnetic fields by the subresistive-scale motions as the suppression mechanism makes heuristic sense, but does not provide an unambiguous verdict on the existence of the dynamo. Indeed, when $\ell_0 \gg \ell_\eta \gg \ell_\nu$, the dominant

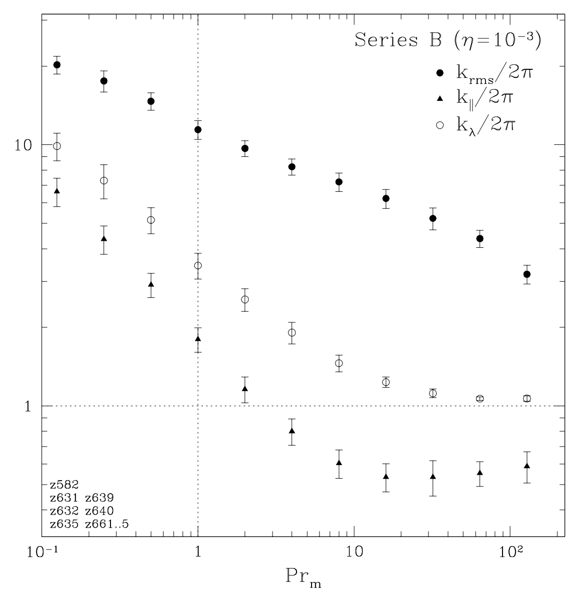

FIG. 3: Characteristic scales for Series B (cf. Fig. 1b): $k_{\rm rms} = \left(\langle|\nabla\mathbf{B}|^2\rangle/\langle B^2\rangle\right)^{1/2}$ is (roughly) the inverse direction-reversal scale, $k_\parallel = \left(\langle|\mathbf{B}\cdot\nabla\mathbf{B}|^2\rangle/\langle B^4\rangle\right)^{1/2}$ is the inverse fold length, and $k_\lambda = \left(\langle|\nabla\mathbf{u}|^2\rangle/\langle u^2\rangle\right)^{1/2}$ is the inverse Taylor microscale (times $\sqrt{5}$) (see Refs. 12, 13 for discussion of these quantities).

contributions to both stretching and mixing are from the resistive scale ℓ_η, and the outcome of their competition is impossible to predict on a heuristic basis. Many

aspects of small-scale dynamo have received successful theoretical treatment in the framework of the Kazantsev model [16], which assumes a Gaussian white-in-time velocity field, $\langle u^i(t,\mathbf{x})u^j(t',\mathbf{x}')\rangle = \delta(t-t')\kappa^{ij}(\mathbf{x}-\mathbf{x}')$. The correlator can be expanded, $\kappa^{ij}(\mathbf{y}) - \kappa^{ij}(0) \sim -y^\alpha$, when $y \sim \ell_B$, the magnetic-field scale. When $\mathrm{Pr_m} \gg 1$, $\ell_B \ll \ell_\nu$, so $\alpha = 2$, corresponding to the spatially smooth viscous-scale eddies. On the other hand, when $\mathrm{Pr_m} \ll 1$, $\ell_B \sim \ell_\eta \gg \ell_\nu$ and magnetic field interacts with rough inertial-range velocities, which must be modeled by $\alpha < 2$. The Kazantsev velocity is not a dynamo if it is too spatially rough, viz., when $\alpha < 1$ [16, 17, 18, 19]. If we set aside fundamental objections to the δ-correlated model and try to compare it to real turbulence, we still face the difficulty of interpreting the δ function. If we write equal-time velocity correlators by replacing the δ function by inverse correlation time $1/\tau_\mathrm{c}$, then the relation between α and the spectral exponent of the turbulence depends on how we choose τ_c. The usual choice for Kolmogorov turbulence is $\tau_\mathrm{c} \sim y^{2/3}$, the eddy-turnover time. Then $\alpha = 4/3 > 1$ and there is dynamo [15][26]. Note that $\mathrm{Rm_c}$ in this case is typically much larger than for $\alpha = 2$ [18, 19, 20]. Although specific values of $\mathrm{Rm_c}$ calculated from the Kazantsev model cannot be considered as quantitative predictions for real turbulence, they appear to point to the second possibility mentioned above (finite but unresolvably large $\mathrm{Rm_c}$). We emphasize that all these results depend on the heuristic choice of τ_c (e.g., if $\tau_\mathrm{c} \sim$ const, $\alpha = 2/3$ and there is no dynamo) and on the universality of the physically nonobvious condition $\alpha > 1$. It is fair to observe that our simulations are still too viscous to have a well-developed Kolmogorov scaling (Fig. 2). Thus, if the existence of the dynamo depends on the exact inertial-range scaling of the velocity field and/or only manifests itself at very large Rm, neither the Kazantsev theory nor simulations at current resolutions can lay claim to a definitive answer. Obviously, the Kazantsev theory also cannot capture any role the intermittency of the velocity field might play and, more generally speaking, it is doubtful that a δ-correlated Gaussian flow is a suitable model of the inertial-range turbulence.

While, as far as we know, ours is the first systematic study of the small-scale dynamo suppression in homogeneous isotropic MHD turbulence with low $\mathrm{Pr_m}$, indications of this effect have been reported in the literature in two previous instances. Dynamo suppression at low $\mathrm{Pr_m}$ was seen by Christensen et al. [21] in their simulations of convection in a rotating spherical shell and by Cattaneo [22] in simulations of Boussinesq convection. These cases of failed low-$\mathrm{Pr_m}$ dynamos in inhomogeneous convection-driven turbulence are likely to be related to the same universal mechanism that made the dynamo inefficient in our simulations. Our key conclusion is that the dynamo suppression is a generic effect unrelated to the particular type of driving or other large-scale features of the system.

The mean-field dynano, which, in contrast, does depend on large-scale features such as helicity and rotation [1], may then be the only type of self-sustained field generation for low-$\mathrm{Pr_m}$ systems. If a mean field is present, it gives rise to a source term $\langle\mathbf{B}\rangle \cdot \nabla\mathbf{u}$ in the induction equation (2) and thus induces small-scale magnetic fluctuations. They have a $k^{-11/3}$ spectrum at $k \gg k_\eta$ [14, 15, 23], which has been seen in the laboratory [8] and in large-eddy simulations [24].

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